European Journal of Combinatorics 32 (2011) 1213–1221

Contents lists available at SciVerse ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Indexes of long zero-sum sequences over cyclic groups

Xiangneng Zeng [a], Pingzhi Yuan [b]

[a] Department of Mathematics, Sun Yat-Sen University, Guangzhou 510275, PR China

[b] School of Mathematics, South China Normal University, Guangzhou 510631, PR China

ARTICLE INFO

Article history:
Received 25 January 2011
Accepted 27 June 2011
Available online 12 August 2011

ABSTRACT

Let G be a cyclic group of order n, and let $S \in \mathcal{F}(G)$ be a zero-sum sequence of length $|S| \geq 2\lfloor n/2 \rfloor + 2$. Suppose that S can be decomposed into a product of at most two minimal zero-sum sequences. Then there exists some $g \in G$ such that $S = (n_1 g) \cdot (n_2 g) \cdot \cdots \cdot (n_{|S|} g)$, where $n_i \in [1, n]$ for all $i \in [1, |S|]$ and $n_1 + n_2 + \cdots + n_{|S|} = 2n$. And we also generalize the above result to long zero-sum sequences which can be decomposed into at most $k \geq 3$ minimal zero-sum sequences.

1. Introduction

Let $\mathbb{N}_0$ denote the set of all nonnegative integers, $\mathbb{Z}$ the set of all integers and $\mathbb{R}$ the set of all real numbers. For $a, b \in \mathbb{R}$, $[a, b] = \{x : a \leq x \leq b, x \in \mathbb{Z}\}$ denotes the set of integers between a and b. For $a \in \mathbb{R}$, $\lfloor a \rfloor = \max\{l \in \mathbb{Z} : l \leq a\}$ and $\lceil a \rceil = \min\{l \in \mathbb{Z} : l \geq a\}$.

Let G be an abelian additive group, in particular, the set of integers $\mathbb{Z}$ is an abelian additive group. Let $\mathcal{F}(G)$ be the free abelian (multiplicative) monoid with basis G. The elements of $\mathcal{F}(G)$ are called *sequences* over G. We write a sequence $S \in \mathcal{F}(G)$ in the form

$$S = g_1 \cdot \cdots \cdot g_l = \prod_{g \in G} g^{\mathsf{v}_g(S)},$$

where $l \in \mathbb{N}_0, g_1, \ldots, g_l \in G$ and $\mathsf{v}_g(S) \in \mathbb{N}_0$. We call $\mathsf{v}_g(S)$ the *multiplicity* of g in S and $|S| = l = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0$ the *length* of S. The unit element $1 \in \mathcal{F}(G)$ is called the *empty sequence*. Denote by $\mathrm{supp}(S) = \{g \in G : \mathsf{v}_g(S) > 0\}$ the *support* of S.

A sequence S_1 is called a *subsequence* of S if $S_1 | S$ in $\mathcal{F}(G)$ (i.e. $\mathsf{v}_g(S_1) \leq \mathsf{v}_g(S)$ for all $g \in G$), and it is called a *proper subsequence* of S if $S_1 | S$, $S_1 \neq 1$ and $S_1 \neq S$. If S_1 is a subsequence of S, we use $S(S_1)^{-1}$ to denote the sequence obtained by deleting the terms of S_1 from S (equivalently, $S = (S(S_1)^{-1}) \cdot S_1$).

E-mail addresses: junevab@163.com (X. Zeng), mcsypz@mail.sysu.edu.cn (P. Yuan).

0195-6698/$ – see front matter
doi:10.1016/j.ejc.2011.06.007

For a sequence S defined above, we define

- $\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g\in G} \mathsf{v}_g(S)g \in G$ the sum of S,
- $\sum(S) = \{\sum_{i\in I} g_i : \emptyset \neq I \subset [1, l]\} = \{\sigma(T) : T|S, T \neq 1\}$ the set of subsums of S, and
- $\sum_n(S) = \{\sigma(T) : T|S, |T| = n\}$ the set of n-term subsums of S.

A sequence S is called

- zero-sum if $\sigma(S) = 0$,
- minimal zero-sum if $S \neq 1$, $\sigma(S) = 0$ and $\sigma(T) \neq 0$ for every proper subsequence $T|S$, and
- zero-sum free if $0 \notin \sum(S)$.

We denote by $\mathcal{B}(G) = \{S \in \mathcal{F}(G) : \sigma(S) = 0\}$ the set of all zero-sum sequences, by $\mathcal{A}(G)$ the set of all minimal zero-sum sequences and by $\mathcal{A}^*(G)$ the set of all zero-sum free sequences in $\mathcal{F}(G)$. Obviously, a zero-sum sequence can be decomposed into a product of some minimal zero-sum sequences (usually the decompositions are not unique).

If S is a zero-sum sequence, we denote by $\mathcal{L}(S)$ the maximum of all l such that $S = S_1 \cdot \cdots \cdot S_l$ with $S_i \in \mathcal{A}(G)$ for all $i \in [1, l]$. In particular, we have $\mathcal{L}(S) = 1$ for any minimal zero-sum sequence S. Note that if $\mathcal{L}(S) = k$ and $S = S_1 \cdot \cdots \cdot S_k$ with $S_i \in \mathcal{A}(G)$ for all $i \in [1, k]$, then $\mathcal{L}(S_1 \cdot \cdots \cdot S_t) = t$ for all $t \in [1, k]$.

Definition 1.1 (*[5, Definition 5.1.1]*)**.** Let G be a cyclic group of order n and $g \in G$ an element with $\mathrm{ord}(g) = n$. For a sequence

$$S = (n_1 g) \cdot (n_2 g) \cdot \cdots \cdot (n_l g), \quad \text{where } l \in \mathbb{N}_0 \text{ and } n_1, \ldots, n_l \in [1, n],$$

we define

$$\|S\|_g = \frac{n_1 + \cdots + n_l}{n}.$$

Obviously, S has sum zero if and only if $\|S\|_g \in \mathbb{N}_0$.

In 2007, Yuan [9] and Savchev and Chen [6] independently proved the following results.

Theorem 1.2. *Let G be a cyclic group of order n and $S \in \mathcal{F}(G)$ a zero-sum free sequence of length*

$$|S| \geq \frac{n+1}{2}.$$

Then there exists some $g \in G$ with $\mathrm{ord}(g) = n$ such that $\|S\|_g < 1$.

Corollary 1.3. *Let G be a cyclic group of order n and $S \in \mathcal{F}(G)$ a minimal zero-sum sequence of length*

$$|S| \geq \frac{n+1}{2} + 1.$$

Then there exists some $g \in G$ with $\mathrm{ord}(g) = n$ such that $\|S\|_g = 1$.

The results give the explicit structure of long minimal zero-sum sequences, which has lots of applications. Let G be a cyclic group of order n, by using the above results, Savchev and Chen [7] gave a structural description of sequences $S \in \mathcal{F}(G)$ with $|S| = n+k$, $\lfloor (n-1)/2 \rfloor \leq k \leq n-2$ and $0 \notin \sum_n(S)$. Gao and Geroldinger [1] showed that $\rho_{2k+1}(G) = kn + 1$, where $\rho_k(G) = \max\{\mathcal{L}(U_1 \cdot \cdots \cdot U_k) : U_i \in \mathcal{A}(G), i \in [1, k]\}$. Gao, et al. [2] proved the following theorem.

Theorem 1.4 (*[2]*)**.** *Let G be a cyclic group of order n and $S \in \mathcal{F}(G)$ a sequence of length $|S| = n$. Suppose that all the zero-sum subsequences of S have the same length. Then $|\mathrm{supp}(S)| \leq 2$.*

For more related problems of the structure theory, the interested readers may see the very recent papers [3,4].

In the present paper, we obtain the structure of long zero-sum sequences, which generalizes Theorem 1.2 and Corollary 1.3.

Theorem 1.5. *Let G be a cyclic group of order n and $S \in \mathcal{F}(G)$ a zero-sum sequence with $\mathcal{L}(S) = k \geq 2$ and $|S| \geq k\lfloor n/2 \rfloor + 2$. Then there exists some $g \in G$ with $\mathrm{ord}(g) = n$ such that $\|S\|_g = k$.*

Note that if S is a zero-sum sequence with $\mathcal{L}(S) = k$, then $\|S\|_g \geq k$ for any $g \in G$ with $\mathrm{ord}(g) = n$. Therefore we only need to prove the converse.

The restriction $|S| \geq k\lfloor n/2 \rfloor + 2$ is sharp in view of the following examples.

$$S = \begin{cases} ((2g)^{n/2-1}(3g)(-g)) \cdot ((2g)^{n/2})^{k-1} & \text{for even } n \geq 6, \\ ((2g)^{(n-5)/2}(3g)^2(-g)) \cdot ((2g)^{(n-3)/2}(3g))^{k-1} & \text{for odd } n \geq 9, \end{cases}$$

where g is an element of G with $\mathrm{ord}(g) = n$. For more examples, see [8].

The paper is organized as follows. In Section 2, we give some results on behaving sequences and strongly behaving sequences. In Section 3, an intermediate theorem is proven, where the main idea is shown. In the last section, the proofs of Theorem 1.5 are presented.

2. Behaving and strongly behaving sequences

Definition 2.1 (*[6]*)**.** A positive integer sequence S with sum $\sigma(S) = n$ is called behaving if the set of subsums $\sum(S) = [1, n]$.

The definition of the behaving sequence is important to the proof of Theorem 1.2. We have

Lemma 2.2 (*[6, Proposition 4]*)**.** *A sequence $S = s_1 \cdot \cdots \cdot s_k$ with positive integer terms in nondecreasing order $s_1 \leq \cdots \leq s_k$ is behaving if and only if*

$$s_1 = 1 \quad \textit{and} \quad s_{i+1} \leq 1 + s_1 + \cdots + s_i \quad \textit{for all } i \in [1, k-1].$$

In particular, $\sigma(S) = s_1 + \cdots + s_k \geq 2s_k - 1$.

For the proofs of our main results, we need the following definition for strongly behaving sequences.

Definition 2.3. A positive integer sequence S with sum $\sigma(S) = n$ is called strongly behaving if for any $x \in [1, n]$, there exists a subsequence $T|S$ such that T is a behaving sequence with sum $\sigma(T) = x$.

Obviously, if S is strongly behaving, then S is behaving. We also have

Lemma 2.4. *A sequence $S = s_1 \cdot \cdots \cdot s_k$ with positive integer terms in nondecreasing order $s_1 \leq \cdots \leq s_k$ is strongly behaving if and only if*

$$s_1 = 1 \quad \textit{and} \quad 2s_{i+1} \leq 2 + s_1 + \cdots + s_i \quad \textit{for all } i \in [1, k-1].$$

Proof. Let $n = \sigma(S) = s_1 + s_2 + \cdots + s_k$. Suppose that S is strongly behaving, then $\sum(S) = [1, n]$. Obviously, $s_1 = 1$, so it is sufficient to show that $2s_{i+1} \leq 2 + s_1 + \cdots + s_i$ for all $i \in [1, k-1]$.

Suppose to the contrary that $2s_{i+1} \geq 3 + s_1 + \cdots + s_i$ for some $i \in [1, k-1]$. Consider the integer $x = 2s_{i+1} - 2 \in [1, n]$. Since S is strongly behaving, there exists a behaving subsequence $T|S$ such that $\sigma(T) = x$. Since $s_1 + \cdots + s_i \leq 2s_{i+1} - 3 < x$, we have $s_j|T$ for some $j \in [i+1, k]$. Since $x = 2s_{i+1} - 2 < s_{j_1} + s_{j_2}$ for all $j_1, j_2 \in [i+1, k]$, we have $T = S_1 s_j$ for some subsequence $S_1|s_1 \cdot \cdots \cdot s_i$. Now we have $\sigma(S_1) = x - s_j \leq s_j - 2$, which contradicts that $T = S_1 s_j$ is behaving.

Conversely, we will prove the sufficiency. The proof is by induction on the length $|S| = k$. The basic case $k = 1$ is trivial. Suppose that we have shown that S' is strongly behaving with length $|S'| < k$. For $S = s_1 \cdot \cdots \cdot s_k$, we apply the induction hypothesis to $S' = s_1 \cdot \cdots \cdot s_{k-1}$ and obtain that S' is strongly behaving. Let $x \in [1, n]$. If $x \leq s_1 + \cdots + s_{k-1}$, then there exists a behaving subsequence $T|S'$ with $\sigma(T) = x$. If $x \geq 1 + s_1 + \cdots + s_{k-1} \geq 2s_k - 1$, we have $x - s_k \leq n - s_k = s_1 + \cdots + s_{k-1}$, and so there is a behaving subsequence $T'|S'$ with $\sigma(T') = x - s_k$. Since $x - s_k \geq s_k - 1$, $T's_k$ is the desired behaving subsequence. This completes the proof of the sufficiency. We are done. $\square$

For the cyclic group of finite order, we have similar definitions for behaving and strongly behaving sequences. In this case, some authors called them smooth sequences instead of behaving ones.

Definition 2.5. Let G be a cyclic group of order n and $S \in \mathcal{F}(G)$ a sequence over G. We say S is (strongly) behaving with respect to $g \in G$ if $S = (a_1 g)(a_2 g)\cdots(a_k g)$ with $1 \le a_i \le n$ such that the positive integer sequence $a_1 a_2 \cdots a_k$ is (strongly) behaving.

Lemma 2.6. *Let $k, n \in \mathbb{N}$ be positive integers, and let $S = s_1 \cdot \cdots \cdot s_k$ be a positive integer sequence such that*

$$s_1 \le \cdots \le s_k, \qquad \sigma(S) = n \quad \text{and} \quad k \ge \frac{n+1}{2}.$$

Then S has one of the following forms.

(i) $S' = s_1 \cdot \cdots \cdot s_{k-1}$ *is strongly behaving.*
(ii) $S = 1^{(n-3)/2} \cdot (\frac{n+3}{4})^2$, *where $n \equiv 1 (\mathrm{mod}\ 4)$ and $k = (n+1)/2$.*
(iii) $S = 1 \cdot 2^{(n-1)/2}$, *where n is odd and $k = (n+1)/2$.*

In particular, $\sum(S) = [1, n]$ in all three cases.

Proof. We first show that $s_1 = 1$. Otherwise, $n = s_1 + \cdots + s_k \ge 2k \ge n+1$, a contradiction.

If $s_{k-1} = 1$, then S has the form (i), and we are done.

Now we consider the case $s_{k-1} > 1$. Let $u \in \mathbb{N}$ denote the maximal index with $s_u = 1$. Obviously, $u \le k-2$.

If $2s_{k-1} \le 2 + u$, since $2s_i = 2 \le 2 + s_1 + \cdots + s_{i-1}$ for all $i \in [2, u]$ and $2s_i \le 2s_{k-1} \le u + 2 \le 2 + s_1 + \cdots + s_{i-1}$ for all $i \in [u+1, k-1]$, we obtain that $S' = s_1 \cdot \cdots \cdot s_{k-1}$ is strongly behaving by Lemma 2.4. Therefore, we may assume that $2s_{k-1} \ge u + 3$.

An easy calculation shows

$$\begin{aligned} n &= s_1 + s_2 + \cdots + s_k \\ &\ge u + 2(k-2-u) + s_{k-1} + s_k, \end{aligned}$$

so $s_{k-1} + s_k \le u + (n+4-2k)$, where the equality holds if and only if $u = k-2$ or $s_{u+1} = \cdots = s_{k-2} = 2$. Note that $u+3 \le 2s_{k-1} \le s_{k-1} + s_k \le u + (n+4-2k) \le u+3$, it follows that $s_k = s_{k-1} = (u+3)/2$ and $k = (n+1)/2$, in particular, $s_{k-1} + s_k = u + (n+4-2k)$, which implies that $u = k-2$ or $s_{u+1} = \cdots = s_{k-2} = 2$.

If $u = k-2$, then S has the form (ii).

If $s_{u+1} = \cdots = s_{k-2} = 2$ and $u = 1$, then S has the form (iii). If $u \ge 2$, then $s_k = s_{k-1} \ge 3 = s_{k-2} + 1$. Since $2s_i = 2 \le 2 + s_1 + \cdots + s_{i-1}$ for all $i \in [2, u]$, $2s_i \le 2s_{k-1} - 2 = u + 1 < 2 + s_1 + \cdots + s_{i-1}$ for all $i \in [u+1, k-2]$ and $2s_{k-1} = u + 3 < 2 + s_1 + \cdots + s_u + \cdots + s_{k-2}$, we obtain that $S' = s_1 \cdot \cdots \cdot s_{k-1}$ is strongly behaving.

Now we prove $\sum(S) = [1, n]$, i.e. S is behaving. The cases when S has forms (ii) and (iii) are obvious. If S has the form (i), we have $s_k = n - \sigma(S') \le (n+1)/2 \le 1 + \sigma(S')$, it follows that S is behaving by Lemma 2.2.

This completes the proof of the lemma. $\square$

3. An intermediate theorem

In the section, we prove an intermediate theorem, where the main idea of the paper is presented.

Theorem 3.1. *Let G be a cyclic group of order $n > 1$ and $S \in \mathcal{F}(G)$ a zero-sum sequence with $|S| \ge 2\lfloor n/2 \rfloor + 2$ and $\mathcal{L}(S) = 2$. Let $g \in G$ be a non-zero element with* $\mathrm{ord}(g) = n$. *Suppose S has a decomposition $S = S_1 S_2$, where*

$$S_1 = (a_1 g) \cdot \cdots \cdot (a_t g), \quad 1 \le a_1 \le \cdots \le a_t < n,\ a_1 + \cdots + a_t = n,\ t \ge n/2 + 1$$

and

$$S_2 = (b_1 g) \cdot \cdots \cdot (b_u g), \quad 1 \le b_1 \le \cdots \le b_u < n,\ u = |S| - t \ge 2.$$

Then $b_1 + b_2 + \cdots + b_u = n$.

Proof. Since $\mathcal{L}(S)=2$, S_1 and S_2 are both minimal zero-sum sequences. Let $m=a_1+a_2+\cdots+a_{t-1}$, then $n > m \geq t-1$. Let $S_1'=(a_1g)\cdot\cdots\cdot(a_{t-1}g)=S_1(a_tg)^{-1}$. By Lemma 2.6, the positive integer sequence $a_1\cdot\cdots\cdot a_{t-1}$ is strongly behaving and so for any $x\in[1,m]$, there exists a behaving subsequence $T|S_1'$ with $\sigma(T)=xg$.

Suppose to the contrary that $b_1+\cdots+b_u=\gamma n$ for some integer $\gamma\geq 2$, then $b_1+\cdots+b_{u-1}=\gamma n-b_u>n$.

Let $v\in\mathbb{N}_0$ denote the maximal index with $b_v\leq m$ (if $b_1>m$, set $v=0$). If $v\geq u-1$, then $b_{u-1}\leq m$. Let $i\in[1,u-1]$ be such that $b_1+\cdots+b_{i-1}<n<b_1+\cdots+b_i$. Since $b_i\leq b_{u-1}\leq m$, there is a behaving subsequence $T|S_1'$ with $\sigma(T)=b_ig$. Since T is behaving, then T has a decomposition $T=T_1T_2$ with $\sigma(T_1)=(n-(b_1+\cdots+b_{i-1}))g$ and $T_2=T(T_1)^{-1}$, and thus we obtain a decomposition

$$S=((b_1g)\cdot\cdots\cdot(b_{i-1}g)T_1)\cdot(T_2(b_{i+1}g)\cdot\cdots\cdot(b_ug))\cdot(S_1T^{-1}(b_ig)),$$

where $(b_1g)\cdot\cdots\cdot(b_{i-1}g)T_1, T_2(b_{i+1}g)\cdot\cdots\cdot(b_ug)$ and $S_1T^{-1}(b_ig)$ are all nonempty zero-sum sequences, which contradicts $\mathcal{L}(S)=2$. Therefore $v<u-1$ and in particular $b_{u-1}>m$.

For every $j_1,j_2\in[v+1,u]$ and $j_1\neq j_2$, we have $b_{j_1}+b_{j_2}\geq 2(m+1)$. If n is even, then $m\geq t-1\geq n/2$ and $b_{j_1}+b_{j_2}\geq n+2$, while if n is odd, then $m\geq(n+1)/2$ and $b_{j_1}+b_{j_2}\geq n+3$. If $b_{j_1}+b_{j_2}\geq n+m$, then $2n-(b_{j_1}+b_{j_2})\leq 2n-(n+m)=n-m\leq m$ and so there is a behaving subsequence $T|S_1'$ with $\sigma(T)=(2n-(b_{j_1}+b_{j_2}))g=(n-b_{j_1})g+(n-b_{j_2})g$ and T has a decomposition $T=T_1T_2$ with $\sigma(T_1)=(n-b_{j_1})g$ and $\sigma(T_2)=(n-b_{j_2})g$. Now we obtain a decomposition

$$S=((b_{j_1}g)T_1)\cdot(T_2(b_{j_2}g))\cdot(S(T(b_{j_1}g)(b_{j_2}g))^{-1}),$$

where $(b_{j_1}g)T_1$, $T_2(b_{j_2}g)$ and $S(T(b_{j_1}g)(b_{j_2}g))^{-1}$ are all nonempty zero-sum sequences, which contradicts $\mathcal{L}(S)=2$. Therefore, for every $j_1,j_2\in[v+1,u]$ and $j_1\neq j_2$, $b_{j_1}+b_{j_2}\in[n+2,n+m-1]$ when n is even or $b_{j_1}+b_{j_2}\in[n+3,n+m-1]$ when n is odd.

We pair the terms of the positive integer sequence $b_{v+1}b_{v+2}\cdots b_{u-1}$. Let $r=\lfloor(u-v-1)/2\rfloor$ and $c_i=b_{v+(2i-1)}+b_{v+2i}-n$ for all $i\in[1,r]$. Notice that when $u-v$ is even, b_{u-1} is left alone. Also notice that $c_i\in[2,m-1]$ when n is even or $c_i\in[3,m-1]$ when n is odd and that $b_{v+(2i-1)}g+b_{v+2i}g=c_ig$ in G.

If $b_u+(b_1+\cdots+b_v)+(c_1+\cdots+c_r)>n$, then there is some $i\in[1,v]$ such that $b_u+b_1+\cdots+b_{i-1}<n<b_u+b_1+\cdots+b_i$ or $j\in[1,r]$ such that $b_u+(b_1+\cdots+b_v)+(c_1+\cdots+c_{j-1})<n<b_u+(b_1+\cdots+b_v)+(c_1+\cdots+c_j)$. If the former inequalities hold, then there is a behaving subsequence $T|S_1'$ with $\sigma(T)=b_ig$ and T has a decomposition $T=T_1T_2$ with $\sigma(T_1)=(n-(b_u+b_1+\cdots+b_{i-1}))g$. Hence we obtain a decomposition

$$S=((b_ug)(b_1g)\cdot\cdots\cdot(b_{i-1}g)T_1)\cdot(T_2(b_{i+1}g)\cdot\cdots\cdot(b_{u-1}g))\cdot(S_0),$$

where S_0 is the remaining terms of S, which contradicts $\mathcal{L}(S)=2$. If the latter inequalities hold, then there is a behaving subsequence $T|S_1'$ with $\sigma(T)=c_jg$. Similarly, $T=T_1T_2$ with $\sigma(T_1)=(n-b_u-(b_1+\cdots+b_v)-(c_1+\cdots+c_{j-1}))g$ and

$$S=((b_ug)(b_1g)\cdot\cdots\cdot(b_{v+2j-2}g)T_1)\cdot(T_2(b_{v+2j-1}g)\cdot\cdots\cdot(b_{u-1}g))\cdot(S_0),$$

again a contradiction. Therefore $b_u+(b_1+\cdots+b_v)+(c_1+\cdots+c_r)\leq n$.

If $u-v=2$, then $b_u+b_1+\cdots+b_{u-2}+b_{u-1}\leq n+b_{u-1}<2n$, which contradicts that $b_1+\cdots+b_u=\gamma n$ for some integer $\gamma\geq 2$. Therefore $u-v\geq 3$ and $r\geq 1$.

We divide the remaining proof into two cases according to whether n is even or not.

Case 1: n is even, then $|S|=t+u\geq n+2$. If $u-v$ is even, then $r=(u-v-2)/2$ and b_{u-1} is left alone when we do the pairing. Hence

$$\begin{aligned} n &> b_u+(b_1+\cdots+b_v)+(c_1+\cdots+c_r)\\ &\geq m+1+v+2r\\ &= m+u-1\\ &\geq t+u-2\\ &\geq n, \end{aligned}$$

a contradiction. If $u-v$ is odd, then $r=(u-v-1)/2$ and b_{u-1} is not left alone. And so

$$\begin{aligned} n &\geq b_u+(b_1+\cdots+b_v)+(c_1+\cdots+c_r)\\ &\geq m+1+v+2r\\ &= m+u\\ &\geq t+u-1\\ &\geq n+1, \end{aligned}$$

which is again a contradiction.

Case 2: n is odd, then $|S|=t+u\geq n+1$. Recall that $c_i\geq 3$ for all $i\in[1,r]$ and that $u-v\geq 3$. If $u-v$ is even, then $r=(u-v-2)/2\geq 1$ and b_{u-1} is left alone when we do the pairing. Then

$$\begin{aligned} n &> b_u+(b_1+\cdots+b_v)+(c_1+\cdots+c_r)\\ &\geq m+1+v+3r\\ &= m+u-1+r\\ &\geq t+u-1\\ &\geq n, \end{aligned}$$

a contradiction. If $u-v$ is odd, then $r=(u-v-1)/2\geq 1$ and b_{u-1} is not left alone. And so

$$\begin{aligned} n &\geq b_u+(b_1+\cdots+b_v)+(c_1+\cdots+c_r)\\ &\geq m+1+v+3r\\ &= m+u+r\\ &\geq t+u\\ &\geq n+1, \end{aligned}$$

which is again a contradiction. This completes the proof. □

4. Proofs of the main theorem

Now we prove the main theorem of the paper. First, we show the case when $\mathcal{L}(S)=2$.

Proof. Since $\mathcal{L}(S)=2$, every proper zero-sum subsequence is minimal zero-sum. Without loss of generality, we may assume that $n>2$.

First we prove the case when n is even. If S has a proper zero-sum subsequence S_1 with $|S_1|\geq n/2+2$, then Corollary 1.3 shows that there is some $g\in G$ such that $\mathrm{ord}(g)=n$ and $\|S_1\|_g=1$ and then Theorem 3.1 implies that $\|S\|_g=2$. If S has a proper zero-sum subsequence S_1 with $|S_1|\leq n/2$, then $|S(S_1)^{-1}|\geq n/2+2$, and so we can consider the subsequence $S(S_1)^{-1}$ and get the theorem.

If every proper zero-sum subsequence T satisfies that $n/2<|T|<n/2+2$, then $|T|=n/2+1$, and thus $|S|=n+2$. Choose an arbitrary subsequence $S_0|S$ of length $|S_0|=n$, then we can apply Theorem 1.4 to S_0 and obtain that $|\mathrm{supp}(S_0)|\leq 2$, hence $|\mathrm{supp}(S)|\leq 2$ as S_0 is chosen arbitrarily. Set $S=g^uh^v$ with $u\geq v$ and $g,h\in G$. It is obvious that $n/2<u<n$. If $\mathrm{ord}(g)<n$, then $g^{\mathrm{ord}(g)}$ is a proper zero-sum of subsequence of length $\mathrm{ord}(g)\leq n/2$, a contradiction. Hence $\mathrm{ord}(g)=n$ and we may assume that $h=lg$ for some $l\in[2,n-1]$. If $\|S\|_g=(u+vl)/n\leq 2$, we are done. Thus we may assume $\|S\|_g=(u+vl)/n\geq 3$, which implies that $vl>2n$. Let $n=bl+r$ for some $b\in\mathbb{N}$ and $r\in[0,k-1]$, then g^rh^b is a proper zero-sum subsequence of S, and we have

$$\begin{cases} r+lb=n\\ r+b=n/2+1. \end{cases}$$

It follows that $(l-2)b=r-2\leq l-3$, a contradiction.

Now we deal with the case when n is odd. If S has a proper zero-sum subsequence of length $\geq(n+3)/2$ or $\leq(n-1)/2$, then the proof is similar to the case when n is even.

If every proper zero-sum subsequence T satisfies that $(n-1)/2 < |T| < (n+3)/2$, then $|T| = (n+1)/2$, and thus $|S| = n+1$. A similar argument shows that $S = g^u h^v$, where $u \geq v$, $g, h \in G$, $\mathrm{ord}(g) = n$, $h = lg$ for some $l \in [2, n-1]$ and $vl > 2n$. Let $n = bl + r$ for some $r \in [0, l-1]$, similarly we have

$$\begin{cases} r + lb = n \\ r + b = (n+1)/2. \end{cases}$$

It follows $(l-2)b = r - 1 \leq l - 2$ and thus $l = (n+1)/2, r = (n-1)/2, b = 1$. Therefore $S = g^u h^v = (2h)^u h^v$ and $\|S\|_h = (2u+v)/n < 3$, which implies $\|S\|_h = 2$.

This completes the proof of the theorem for $\mathcal{L}(S) = 2$. □

To prove Theorem 1.5 for $k \geq 3$, we need two more lemmas.

Lemma 4.1. *Let $a, b \in \mathbb{N}_0$ be nonnegative integers, $n \geq 3$ an odd integer and $t \in \mathbb{N}$ a positive integer. Suppose*

$$a\frac{n+1}{2} + b = tn.$$

Then $a + b \geq (n+1)/2$ and the equality occurs if and only if $a = 1, b = (n-1)/2$.

Proof. If $b = 0$, then a is a multiple of n, and so $a + b \geq n > (n+1)/2$.

If $b > 0$, let $a = 2k + r$, where $k \in \mathbb{N}_0$ and $r \in [0, 1]$, then $(t-k)n = k + b + r(n+1)/2$. If $r = 0$, then $k + b = (t-k)n \geq n > (n+1)/2$, and so $a + b \geq k + b > (n+1)/2$. If $r = 1$, then $k+b = (t-k)n - (n+1)/2 \geq (n-1)/2$. And so $a+b = 2k+r+b \geq 1+k+(n-1)/2 \geq (n+1)/2$, where the equality holds if and only if $k = 0$, $a = 1$ and $b = (n-1)/2$. □

Lemma 4.2. *Let G be a cyclic group of odd order n and $S \in \mathcal{B}(G)$ such that $\mathcal{L}(S) = k \geq 3$ and $|S| \geq k(n-1)/2 + 2$. Let $g \in G$ be an element of order $\mathrm{ord}(g) = n$. Suppose that S has a decomposition $S = S_0 \cdot \cdots \cdot S_{k-1}$ such that $S_0 = (a_1 g) \cdot \cdots \cdot (a_u g) \in \mathcal{A}(G)$, $1 \leq a_1 \leq \cdots \leq a_u \leq n$, $u \leq (n+1)/2$ and $S_1 = \cdots = S_{k-1} = g^{(n-1)/2} \cdot ((n+1)/2\, g)$. Then one of the following cases holds.*

(i) *S has another decomposition $S = S'_0 \cdot \cdots \cdot S'_{k-1}$ such that $S'_i \in \mathcal{A}(G)$ for all $i \in [0, k-1]$ and $|S'_0| > (n+1)/2$.*
(ii) *$\|S_0\|_g = 1$.*

Proof. Since $|S| \geq k(n-1)/2 + 2$, $|S_0| = |S| - (|S_1| + \cdots + |S_{k-1}|) \geq k(n-1)/2 - (k-1)(n+1)/2 = (n+1)/2 - (k-2)$.

Suppose there is an index subset $I = \{i_1, i_2, \ldots, i_{k-1}\} \subset [1, u]$ such that $|I| = k-1$ and $a_i \notin \{1, (n+1)/2\}$ for any $i \in I$. For every $i \in I$, there is a subsequence $T_i | g^{(n-1)/2} \cdot ((n+1)/2\, g)$ such that $|T_i| \geq 2$ and $\sigma(T_i) = a_i g$. In particular, $(n+1)/2\, g \in \mathrm{supp}(T_i)$ when $a_i > (n+1)/2$. Consequently,

$$S = S'_0 S'_1 \cdot \cdots \cdot S'_{k-1},$$

where

$$S'_0 = S_0 \cdot \left(\prod_{i \in I} (a_i g)\right)^{-1} \cdot \left(\prod_{i \in I} T_i\right)$$

and

$$S'_j = S_j \cdot (T_{i_j})^{-1} \cdot (a_{i_j} g) \quad \text{for all } i_j \in I.$$

It follows that $|S'_0| \geq |S_0| - (k-1) + 2(k-1) > (n+1)/2$, which shows that case (i) holds.

Let $I = \{i \in [1, u] : a_i \neq 1 \text{ and } a_i \neq (n+1)/2\}$. By the above argument, we can restrict to the case that $|I| \leq k-2$. Similarly as above, replace $a_i g$ by T_i for every $i \in I$ and get a new decomposition

$$S = S'_0 \cdot \cdots \cdot S'_{k-1},$$

where

$$S_0' = S_0 \cdot \left(\prod_{i\in I}(a_i g)\right)^{-1} \cdot \left(\prod_{i\in I} T_i\right),$$

$$S_j' = S_j \cdot (T_{i_j})^{-1} \cdot (a_{i_j} g) \quad \text{for all } j \in [1, |I|]$$

and

$$S_j' = S_j \quad \text{for all } j \in [|I| + 1, k - 1].$$

Hence $S_0' = g^{\alpha}((n+1)/2g)^{\beta}$ for some $\alpha, \beta \in \mathbb{N}_0$. If $\beta \neq 1$, then $|S_0'| = \alpha + \beta > (n+1)/2$ by Lemma 4.1, and so (i) holds. If $\beta = 1$, then $S_0' = g^{(n-1)/2} \cdot ((n+1)/2\, g)$. Since $\|S_j'\|_g = 1$ for all $j \in [0, k-1]$, we have $\|S\|_g = k$ and thus $\|S_0\|_g = \|S\|_g - (\|S_1\|_g + \cdots + \|S_{k-1}\|_g) = 1$, which is case (ii). The lemma is proved. $\square$

Now we proceed with the proof of Theorem 1.5 for $\mathcal{L}(S) = k \geq 3$.

Proof. The proof is by induction on k. The basic case when $k = 1$ is Corollary 1.3, while the case when $k = 2$ has been proven at the beginning of the section. So it is sufficient to prove the cases when $k \geq 3$. Suppose that the theorem holds for any $S' \in \mathcal{B}(G)$ with $\mathcal{L}(S') < k$.

Let S be a zero-sum sequence with $\mathcal{L}(S) = k$ and have a decomposition $S = S_1 \cdot \cdots \cdot S_k$, where $1 \leq |S_1| \leq \cdots \leq |S_k|$ and $S_i \in \mathcal{A}(G)$ for all $i \in [1, k]$. Let $s_i = |S_i|$ for all $i \in [1, k]$. We may choose a decomposition such that s_k is maximal among all such decompositions. Obviously, $s_k \geq \lfloor n/2 \rfloor + 1$. Notice that $S' = S_2 S_3 \cdots S_k$ is a zero-sum sequence with $|S'| \geq (k-1)\lfloor n/2 \rfloor + 2$ and $\mathcal{L}(S) = k - 1$, thus the induction hypothesis implies that $\|S'\|_g = k - 1$ for some $g \in G$ with $\mathrm{ord}(g) = n$. It is sufficient to show $\|S_1\|_g = 1$. We divide the proof into two cases.

Case 1: $s_k \geq \lceil n/2 \rceil + 1$.

If $s_1 \leq 2$, then $\|S_1\|_g < 2$, which implies that $\|S_1\|_g = 1$ and $\|S\|_g = \|S_1\|_g + \|S'\|_g = k$. Therefore we may assume that $|S_1| \geq 3$.

If $s_1 + s_k \geq 2\lfloor n/2 \rfloor + 2$, then Theorem 3.1 implies that $\|S_1\|_g = 1$ and thus $\|S\|_g = \|S'\|_g + \|S_1\|_g = k$.

If $s_1 + s_k \leq 2\lfloor n/2 \rfloor + 1$, then $s_{k-1} \geq \lfloor n/2 \rfloor + 1$, otherwise $|S| = (s_1 + s_k) + s_2 + \cdots + s_{k-1} < k\lfloor n/2 \rfloor + 2$. Let $S_1 = (a_1 g) \cdot \cdots \cdot (a_{s_1} g)$, $1 \leq a_1 \leq \cdots \leq a_{s_1} \leq n$, $S_{k-1} = (b_1 g) \cdot \cdots \cdot (b_{s_{k-1}} g)$, $1 \leq b_1 \leq \cdots \leq b_{s_{k-1}} \leq n$, and $S_k = (c_1 g) \cdot \cdots \cdot (c_{s_k}) g$, $1 \leq c_1 \leq \cdots \leq c_{s_k} \leq n$. Since $s_1 \geq 3$, we have $s_k \leq 2\lfloor n/2 \rfloor + 1 - s_1 < n$ and then $c_{s_k} > 1$. By Lemma 2.6, S_{k-1} has three forms, thus we divide the proof into three subcases.

Subcase 1.1: $b_1 b_2 \cdot \cdots \cdot b_{s_{k-1}-1}$ is strongly behaving. Since

$$\begin{aligned} c_{s_k} &= n - (c_1 + c_2 + \cdots + c_{s_k - 1}) \\ &\leq n - (s_k - 1) \\ &\leq n - \lceil n/2 \rceil \\ &\leq s_{k-1} - 1 \\ &\leq b_1 + \cdots + b_{s_{k-1}-1}, \end{aligned}$$

there exists a behaving subsequence $T | S_{k-1}$ with $\sigma(T) = c_{s_k} g$. In particular, $|T| \geq 2$. It follows that $S_{k-1} S_k = (S_{k-1} T^{-1}(c_{s_k} g)) \cdot (S_k (c_{s_k} g)^{-1} T)$, which contradicts that $|S_k| = s_k$ is maximal.

Subcase 1.2: $S_{k-1} = g^{(n-3)/2}((n+3)/4\, g)^2$, where $n \equiv 1 (\mathrm{mod}\ 4)$ and $s_{k-1} = (n+1)/2$. If $c_{s_k} \neq (n+3)/4$, then there exists a subsequence $T | S_{k-1}$ with $\sigma(T) = c_{s_k} g$. Since $c_{s_k} \notin \{1, (n+3)/4\}$ and $|T| \geq 2$, replacing $c_{s_k} g$ by T we derive a contradiction with that s_k is maximal. If $c_{s_k} = (n+3)/4$ and $(n-3)/2 \geq (n+3)/4$, then replace $c_{s_k} g$ by $g^{(n+3)/4}$ and get a contradiction. If $c_{s_k} = (n+3)/4$ and $(n-3)/2 < (n+3)/4$, then $n = 5$ and $s_1 \leq (s_1 + s_k)/2 \leq 5/2$, again a contradiction.

Subcase 1.3: $S_{k-1} = g \cdot (2g)^{(n-1)/2}$, where n is odd and $s_{k-1} = (n+1)/2$. If $c_{s_k} > 2$, then there exists a subsequence $T | S_{k-1}$ such that $|T| \geq 2$ and $\sigma(T) = c_{s_k} g$, replacing $c_{s_k} g$ by T we get a contradiction. If $c_{s_k} = 2$, then $S_k = g^{\alpha}(2g)^{\beta}, \alpha, \beta \in \mathbb{N}_0$. Since $s_k \geq \lceil n/2 \rceil + 1 = (n+3)/2$, then $\alpha \geq 2$ and S_k is strongly behaving with respect to g. If $\|S_1\|_g > 1$, let $i \in [1, s_1]$ be such that $a_1 + \cdots + a_{i-1} < n < a_1 + \cdots + a_i$. There exists a behaving subsequence $T|S_k$ with $\sigma(T) = a_i g$ and a decomposition $T = T_1T_2$ with $\sigma(T_1) = (n - (a_1 + \cdots + a_{i-1}))g$. It follows that $S_1S_k = ((a_1g) \cdot \cdots \cdot (a_{i-1}g)T_1) \cdot (T_2(a_{i+1}g) \cdot \cdots \cdot (a_{s_1}g)) \cdot (S_kT^{-1}(a_ig))$, which contradicts $\mathcal{L}(S_1S_k) = 2$. Therefore $\|S_1\|_g = 1$ and $\|S\|_g = k$.

Case 2: $s_k = (n+1)/2$. Let $r = (n+1)/2$ and let $u \in \mathbb{N}$ denote the maximal index with $s_u < r$ (if $s_1 = r$, set $u = 1$). Note that $1 \leq u \leq k-2$. Choose arbitrarily $j_1, j_2 \in [u+1, k]$ with $j_1 \neq j_2$. We claim that $\operatorname{supp}(S_{j_1}) = \operatorname{supp}(S_{j_2})$. Otherwise, without loss of generality, we may assume that an element $h \in G$ satisfies that $h \in \operatorname{supp}(S_{j_1})$ but $h \notin \operatorname{supp}(S_{j_2})$. By Lemma 2.6, there exists a subsequence $T|S_{j_2}$ such that $|T| \geq 2$ and $\sigma(T) = h$. Replace h by T, we derive a contradiction with that s_k is maximal. We divide the remaining proof into three subcases.

Subcase 2.1: $S_k = (c_1g) \cdot \cdots \cdot (c_rg)$ such that $1 \leq c_1 \leq \cdots \leq c_r \leq n$ and that the positive integer sequence $c_1 \cdot \cdots \cdot c_{r-1}$ is strongly behaving. Choose arbitrarily $j \in [u+1, k-1]$ and suppose $S_j = (b_1g) \cdot \cdots \cdot (b_rg)$ with $1 \leq b_1 \leq \cdots \leq b_r \leq n$. If $c_r \leq (n-1)/2$, then $b_r \leq n - (b_1 + \cdots + b_{r-1}) \leq (n+1)/2 \leq n - c_r = c_1 + \cdots + c_{r-1}$. Hence there exists a behaving subsequence $T|S_k$ with $\sigma(T) = b_rg$. In particular, $|T| \geq 2$. Replace b_rg by T and get a contradiction with that s_k is maximal. If $c_r \geq (n+1)/2$, then $n = c_1 + \cdots + c_r \geq r - 1 + (n+1)/2 = n$, and so $S_k = g^{r-1} \cdot (rg)$. By Lemma 4.1 and that $\operatorname{supp}(S_{j_1}) = \operatorname{supp}(S_{j_2})$ for any $j_1, j_2 \in [u+1, k]$, $S_j = g^{r-1} \cdot (rg)$ for any $j \in [u+1, k]$. By Lemma 4.2, either another decomposition yields a greater $|S_k|$, which is a contradiction, or $\|S_i\|_g = 1$ for all $i \in [1, u]$, which yields $\|S\|_g = k$.

Subcase 2.2 $S_k = g^{(n-3)/2}((n+3)/4\,g)^2$. The subcase is the same as the subcase 1.2.

Subcase 2.3 $S_k = g \cdot (2g)^{(n-1)/2}$. Let $h = 2g$, then $S_k = h^{(n-1)/2} \cdot ((n+1)/2\,h)$. The subcase is the same as the subcase 2.1 and we obtain $\|S\|_h = k$.

This completes the proof of the theorem. □

Remark. In the proof above, we show $\|S\|_{2g} = k$ in Subcase 2.3. Indeed, it is also true $\|S\|_g = k$. Therefore, the element g with $\|S\|_g = k$ is not uniquely determined. We give an example. Let G be a cyclic group of odd order n, $g \in G$ of order $\operatorname{ord}(g) = n$ and $h = 2g$. Let $S = (S_0)^k$, where

$$S_0 = h^{(n-1)/2} \cdot ((n+1)/2\,h) = g \cdot (2g)^{(n-1)/2}.$$

Obviously, $\|S\|_g = \|S\|_h = k$.

Acknowledgments

Supported by the NSF of China (No. 10971072) and by the Guangdong Provincial Natural Science Foundation (No. 10152606101000000).

References

[1] W. Gao, A. Geroldinger, On products of k atoms, Monatsh. Math. 156 (2009) 141–157.
[2] W. Gao, Y. Hamidoune, G. Wang, Distinct lengths modular zero-sum subsequences: a proof of Grahams conjecture, J. Number Theory 130 (2010) 1425–1431.
[3] W. Gao, Y. Li, J. Peng, C. Plyley, G. Wang, On the index of sequences over cyclic groups, Acta Arith. 148 (2011) 119–134.
[4] W. Gao, J. Peng, G. Wang, Behaving sequences, J. Combin. Theory Ser. A 118 (2011) 613–622.
[5] A. Geroldinger, Additive group theory and non-unique factorizations, in: A. Geroldinger, I. Ruzsa (Eds.), Combinatorial Number Theory and Additive Group Theory, in: Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, 2009, pp. 1–86.
[6] S. Savchev, F. Chen, Long zero-free sequences in finite cyclic groups, Discrete Math. 307 (2007) 2671–2679.
[7] S. Savchev, F. Chen, Long n-zero-free sequences in finite cyclic groups, Discrete Math. 308 (2008) 1–8.
[8] X. Xia, P. Yuan, Indexes of unsplittable minimal zero-sum sequences of length $\mathbb{I}(C_n) - 1$, Discrete Math. 310 (2010) 1127–1133.
[9] P. Yuan, On the index of minimal zero-sum sequences over finite cyclic groups, J. Combin. Theory Ser. A 114 (2007) 1545–1551.